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Robust extended Kalman filter of discrete-time Markovian jump nonlinear system under uncertain noise

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Abstract

This paper examines the problem of robust extended Kalman filter design for discrete -time Markovian jump nonlinear systems with noise uncertainty. Because of the existence of stochastic Markovian switching, the state and measurement equations of underlying system are subject to uncertain noise whose covariance matrices are time-varying or un-measurable instead of stationary. First, based on the expression of filtering performance deviation, admissible uncertainty of noise covariance matrix is given. Secondly, two forms of noise uncertainty are taken into account: Non-Structural and Structural. It is proved by applying game theory that this filter design is a robust mini-max filter. A numerical example shows the validity of the method.

Keywords: Discrete-time Markovian jump system; Robust Kalman filter; Noise uncertainty

1. Introduction

One of the main issues in control systems is their capability of maintaining an acceptable behavior and meeting some performance requirements even in the presence of abrupt changes in the system dynamics. These changes can be due, for instance, to abrupt environmental disturbances, component failures or repairs, changes in subsystems interconnections, abrupt changes in the operation point for a nonlinear plant etc. Examples of these situations can be found in economic systems, aircraft control systems, control of solar thermal central receivers, robotic manipulator systems, large flexible structures for space stations, etc. [1]. In some cases, these systems can be modeled by a set of discrete-time systems with mode transition given by a Markov chain. This family is known in the specialized literature as a Markovian jump system.

With further study of Markovian jump systems, many achievements have been made in the last decade on stability analysis [2, 3], filtering [4, 5] and controller design [6, 7]. Among the efforts towards filtering, the celebrated Kalman filtering provides an optimal state estimator for the Markovian jump systems with satisfying performance. By assuming the dynamical system is subject to stationary Gaussian input and measurement noise process, the optimal filtering gain could be deduced in terms of coupled algebraic Riccati equations. Based on this, Boukas [8] and Mahmoud [9] gave Kalman filtering equations for continuous-time and discrete-time Markovian jump linear systems with structure uncertainty, respectively. However, in the above referred contributions to filtering problems, all the research work was grounded on one assumption: both the state equation and output measurement are subjected to stationary Gaussian noises so that an optimal filtering gain is obtained based on the exactly-known noise covariance matrix. But this is not the case for Markovian jump systems.

In a practical environment, because of the stochastic switching in Markovian jump systems, which is usually accompanied by sudden change of working

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environment or system dynamics, the statistical characteristics (covariance matrix) of noise may be time-varying instead of stationary. In some cases it is impossible to get the exact measurement value of the noise covariance matrix, which means the noise covariance matrix is uncertain instead of exactly known; for this reason the stochastic noise is so-called "uncertain". As it has been pointed out above, the optimal Kalman filtering gain is deduced in terms of coupled algebraic Riccati equations using noise statistical characteristics. Thus, uncertainty to noise covariance matrix affects the Kalman filtering gain and the estimation of system state, which will ultimately affect the control signals. Obviously, noise uncertainty will decrease system performance and in the worst case lead to system instability. And with larger noise uncertainty, system instability tends to occur with more probability. For these reasons, there must be some limitation (bound) to noise uncertainty so that the whole system can maintain an acceptable behavior. This paper is focusing on how to achieve the maximum bound of noise uncertainty according to system performance requirements.

The rest of this paper is organized as follows: First, some assumptions are given so that the nonlinear jump systems could be modeled as a linear one by local linearization. Second, according to system performance requirements, the maximum upper bound of uncertainty to noise covariance matrix is discussed in two forms: non-structural and structural. Then the analytical solution of maximum bound is obtained by using *Lagrange* method. Finally, the establishment of saddle inequality is proved, and this result shows that this robust extended Kalman filter design is a minimax robust filter. At the end of the paper, an illustrative example is used to show the validity of the mentioned method.

2. Problem description

Throughout the paper, unless otherwise specified, we denote by $(\Omega, F, \{F_t\}_{t\geq 0}, Pr)$, a complete probability space with a filtration $\{F_t\}_{t\geq 0}$ satisfying the usual conditions (i.e., it is right continuous and F_0 contains all p-null sets). Let $|x|$ stand for the usual Euclidean norm for a vector *x*, and $|X|$ denote the Frobenius norm of a matrix X defined by $|X| = \lambda_{max}^{\frac{1}{2}}(XX^T)$, where $\lambda_{max}(\cdot)$ is the maximum eigenvalue of matrix and the superscript T represents transpose. Operator $tr(\cdot)$ denotes the matrix trace and we denote by $X > 0 (\geq 0)$ that matrix X is positive definite (semi-positive definite). Let $\{r_k, k \geq 0\}$ be a discrete-time Markov chain on the probability space taking values in finite state space $S = \{1, 2, \cdots, N\}$ with $P = [p_{ij}]$ the chain generator, an $N \times N$ matrix. The entries $p_{ij}, i, j \in S$ are interpreted as transition rates such that

$$p_{ij} = Pr(r_{k+1} = j \mid r_k = i)$$

Here $p_{ij} \geq 0$ is the transition probability from *i* to *j*. Notice that the total probability axiom imposes

$$\sum_{j=1}^{N} p_{ij} = 1, p_{ij} \geq 0 \quad \forall i \in S$$

Consider the following discrete-time Markovian jump nonlinear system with uncertain noise:

$$x(k+1) = f(x(k), r_k) + \omega^0$$
$$y(k) = h(x(k), r_k) + \upsilon^0 \tag{1}$$

where $x(k) \in \mathbf{R}^n$ is state vector, $y(k) \in \mathbf{R}^m$ is measurement output. $f(\cdot,\cdot) \in \mathbf{R}^n$, $h(\cdot,\cdot) \in \mathbf{R}^m$ are nonlinear vector functions. ω^0, υ^0 are *n*-dimensional and *m*-dimensional white noise and satisfy the following assumption:

Assumption 1. For any given time $s, \tau \geq 0$,

(1) $E[\omega_\tau^0] = 0 \qquad E[\upsilon_\tau^0] = 0$

(2) $Cov[\omega_s^0, \omega_\tau^0] = W^0 \delta_{s,\tau} = (W + \Delta W)\delta_{s,\tau}$, $W \geq 0, \Delta W \geq 0$

(3) $Cov[\upsilon_s^0, \upsilon_\tau^0] = V^0 \delta_{s,\tau} = (V + \Delta V)\delta_{s,\tau}$, $V > 0, \Delta V \geq 0$

(4) $E[\begin{pmatrix} w_s^0 \\ \upsilon_s^0 \end{pmatrix} \cdot (w_\tau^{0T} \quad \upsilon_\tau^{0T})] = \begin{bmatrix} W^0 \delta_{s,\tau} & 0 \\ 0 & V^0 \delta_{s,\tau} \end{bmatrix}$

In Assumption 1, $W^0 \in \mathbf{R}^{n\times n}, V^0 \in \mathbf{R}^{m\times m}$ consist of two parts, where W, V denote the stationary noise covariance matrices and the values are exactly known. $\Delta W, \Delta V$ denote the uncertainty caused by disturbance or time-varying; they are unknown but bounded. $\delta(\cdot,\cdot)$ is a *Dirac* function taking values in $\{0,1\}$.

For the deduction of extend Kalman filter, the following assumption is proposed:

Assumption 2. For any fixed system mode $r_k = i \in S$, the nonlinear vector functions $f(\cdot,\cdot)$, $h(\cdot,\cdot)$ are assumed to satisfy $f(0,i) = h(0,i) = 0$ and

$$| f(x(k)+\sigma,i) - f(x(k),i) - A(i)\sigma | \le | \Delta A(i) || \sigma | \quad (2)$$

$$| h(x(k)+\sigma,i) - h(x(k),i) - C(i)\sigma | \le | \Delta C(i) || \sigma | \quad (3)$$

where $A(i), C(i)$ are Jacobian matrices of $f(\cdot,\cdot), h(\cdot,\cdot)$ and $\Delta A(i), \Delta C(i)$ satisfy

$$\begin{aligned} \Delta A(i) &= H_1(i)F(i)E(i) \\ \Delta C(i) &= H_2(i)F(i)E(i) \end{aligned} \quad (4)$$

Here $H_1(i), H_2(i), E(i), i \in S$ are known constant matrix and $F(i), i \in S$ is unknown matrix satisfying $F^T(i)F(i) \le I$. It is known that with Assumption 2 established [10], the Markovian jump nonlinear system could be transformed to a nominal linear model via local linearization technique:

$$\begin{aligned} x(k+1) &= [A(r_k) + \Delta A(r_k)]x(k) + \omega^0 \\ y(k) &= [C(r_k) + \Delta C(r_k)]x(k) + \upsilon^0 \end{aligned} \quad (5)$$

For simplification, we denote $A(r_k = i)$, $H_1(r_k = i)$, $H_2(r_k = i)$, $E(r_k = i)$, $\Delta A(r_k = i)$, $C(r_k = i)$, $\Delta C(r_k = i)$ by A_i, H_{1i}, H_{2i}, E_i, ΔA_i, C_i, ΔC_i.

Theorem 1 Consider stochastically stable Markovian jump system (1) or (5) and assume the noise is stationary, which means $\Delta W = \Delta V = 0$, the standard extended Kalman filter is as follows [9]:

$$\hat{x}(k+1) = \hat{A}_i \hat{x}(k) + K_i[y(k) - \hat{C}_i \hat{x}(k)] \quad (6)$$

where filtering gain K_i is given by the following coupled Riccati equations:

$$\begin{aligned} \Psi_i &= A_i(\sum_{j=1}^{N} p_{ij}\Psi_j)A_i^T + \varepsilon_i E_i^T E_i + \varepsilon_i H_{1i}H_{1i}^T + W \\ \hat{A}_i &= A_i + (\varepsilon_i H_{1i}H_{1i}^T + W)\Psi_i^{-1} \\ \hat{C}_i &= C_i + \frac{1}{\varepsilon_i} H_{2i}H_{1i}^T\Psi_i^{-1} \\ Q_i &= (\hat{A}_i - K_i\hat{C}_i)(\sum_{j=1}^{N} p_{ij}Q_j)(\hat{A}_i - K_i\hat{C}_i)^T + K_iVK_i^T + W + \varepsilon_i H_{1i}H_{1i}^T \\ K_i &= [\hat{A}_i(\sum_{j=1}^{N} p_{ij}Q_j)\hat{C}_i^T][V + \frac{1}{\varepsilon_i}H_{2i}H_{2i}^T + \hat{C}_i(\sum_{j=1}^{N} p_{ij}Q_j)\hat{C}_i^T]^{-1} \end{aligned} \quad (7)$$

Here matrix $\Psi_i > 0$, $Q_i > 0$ and scalar $\varepsilon_i > 0$ are chosen such that $tr(Q_i)$ reaches the minimum. With the above standard Kalman filter gain (6) adopted, the state estimation error satisfies:

$$E\{(x(k) - \hat{x}(k))^T(x(k) - \hat{x}(k))\} \le \max_{j \in S} tr(Q_j) \quad (8)$$

Define the estimation error performance of standard Kalman filtering as

$$J(K_1, K_2, \cdots, K_N, W, V) = \max_{j \in S} tr(Q_j) \quad (9)$$

According to Theorem 1 and quality of Kalman filtering, if the noise is stationary $(\Delta W = \Delta V = 0)$, the estimation error performance could achieve the minimum value by adopting standard Kalman filtering (6).

However, in practice, the standard Kalman filter may fail with uncertain noise: $\Delta W \ne 0$, $\Delta V \ne 0$; thus, the new covariance matrix of noise is W^0, V^0. If the designer still adopts the former pre-designed Kalman filter gain K_i, the new state estimation error should be Q_i^0, which satisfies:

$$Q_i^0 = (\hat{A}_i - K_i\hat{C}_i)(\sum_{j=1}^{N} p_{ij}Q_j^0)(\hat{A}_i - K_i\hat{C}_i)^T + K_iV^0K_i^T + W^0 + \varepsilon_i H_{1i}H_{1i}^T \quad (10)$$

Therefore, the new estimation performance is

$$J(K_1, K_2, \cdots, K_N, W^0, V^0) = \max_{j \in S} tr(Q_j^0) \quad (11)$$

According to (9) and (11), the deviation of estimation performance yielded by noise uncertainty $(\Delta W, \Delta V)$ can be written as:

$$\begin{aligned} \Delta J(K_1, K_2, \cdots, K_N, \Delta W, \Delta V) &= J(K_1, \cdots, K_N, W^0, V^0) \\ &\quad - J(K_1, \cdots, K_N, W, V) \\ &= \max_{j \in S} tr(Q_j^0) - \max_{j \in S} tr(Q_j) \le r \end{aligned} \quad (12)$$

Thus the problem is: if the designer wants the pre-designed Kalman filter to still meet the performance requirements under uncertain noise $(\Delta W, \Delta V)$, he or she should limit noise uncertainty to a certain bound. As long as $(\Delta W, \Delta V)$ is within this bound, the admissible deviation of estimation performance $\Delta J(K_1, K_2, \cdots, K_N, \Delta W, \Delta V) \le r$ where $r > 0$ is a constant parameter given by the designer according to practical requirement. In the following work, the author sets out to find the corresponding expression of noise uncertainty $(\Delta W, \Delta V)$ and the performance r.

3. Upper bound of noise uncertainty

3.1 Mathematical expression

Combining Eq. (7) and (10), there is

$$\Delta Q_i = (\hat{A}_i - K_i\hat{C}_i)\sum_{j=1}^{N} p_{ij}\Delta Q_j(\hat{A}_i - K_i\hat{C}_i)^T + K_i\Delta VK_i^T + \Delta W \quad (13)$$

Where $\Delta Q_i = Q_i^0 - Q_i$. According to Eq. (13), it is easy to see that $tr(\Delta Q_i)$ is a linear mapping of $(\Delta W, \Delta V)$.

Define a compact convex set as $\Xi = \{(\Delta W, \Delta V) : 0 \le \Delta W \le \Delta W^*, 0 \le \Delta V \le \Delta V^*\}$; thus, the deviation of performance $\Delta J(K_1,K_2,\cdots,K_N,\Delta W,\Delta V)$ is a mapping from Ξ to $\mathbf{R}^1$, and it has the following facts:

Fact 1 For any given $(\Delta W_j,\Delta V_j)\in\Xi, j=1,2$, if $\Delta W_1\le\Delta W_2,\Delta V_1\le\Delta V_2$, there is

$$\Delta J(K_1,K_2,\cdots,K_N,\Delta W_1,\Delta V_1)\le\Delta J(K_1,K_2,\cdots,K_N,\Delta W_2,\Delta V_2)$$

This means the deviation performance is a monotonically increasing function of noise uncertainty.

Fact 2 Define the maximum admissible deviation of estimation performance r as

$$r=\max_{(\Delta W,\Delta V)\in\Xi}\Delta J(K_1,K_2,\cdots,K_N,\Delta W,\Delta V)$$

Thus r could be achieved only by maximum matrix pair $(\Delta W^*,\Delta V^*)$. This fact is deduced according to **Fact 1**, and that Ξ is a compact convex set.

The purpose of the following work is to construct a maximum compact convex set $\Xi^*=\{(\Delta W^*,\Delta V^*)\}$; for any uncertainty $(\Delta W,\Delta V)\in\Xi^*$, inequality (12) is sure to establish, and a mini-max robust filtering is applied to minimize the worst performance under the noise uncertainty.

According to the finite of mode S, inequality (12) is equivalent to

$$tr(Q_i^0)\le r+\max_{j\in S}tr(Q_j)\qquad\forall i\in S\tag{14}$$

Therefore, for each mode $i\in S$, there is

$$tr(\Delta Q_i)=tr(Q_i^0)-tr(Q_i)\le r+\max_{j\in S}tr(Q_j)-tr(Q_i)\tag{15}$$

3.2 Bound of nonstructural uncertainty

Suppose there is no limitation or requirement on the structure of noise uncertainty. Let $|\Delta W|\le a$, $|\Delta V|\le b$; thus

$$0\le\Delta W\le aI_n\qquad 0\le\Delta V\le bI_m$$

According to Fact 2, if the noise uncertainty $(\Delta W,\Delta V)$ reaches maximum aI_n,bI_m, the deviation of estimation performance will reach the maximum value r.

According to (13) and (15), for any system mode $\forall i\in S$, there is

$$atr(D_i)+btr(G_i)\le r+\max_{j\in S}tr(Q_j)-tr(Q_i)\tag{16}$$

where matrix $D_i,G_i>0,i\in S$ satisfies the following equations:

$$D_i=(\hat{A}_i-K_i\hat{C}_i)\sum_{j=1}^N p_{ij}D_j(\hat{A}_i-K_i\hat{C}_i)^T+I_n$$

$$G_i=(\hat{A}_i-K_i\hat{C}_i)\sum_{j=1}^N p_{ij}G_j(\hat{A}_i-K_i\hat{C}_i)^T+K_iK_i^T$$

By the above analysis, the search for an admissible upper bound of nonstructural noise uncertainty $(\Delta W,\Delta V)$ is equal to getting the optimal solution of a,b which satisfies the inequalities:

$$\begin{aligned}&max\qquad a\cdot b\\ &s.t.\ a\cdot tr(D_i)+b\cdot tr(G_i)\le r+\max_{j\in S}tr(Q_j)-tr(Q_i)\\ &a\ge 0\qquad b\ge 0\qquad i\in S\end{aligned}\tag{17}$$

Therefore, the search for an admissible upper bound of $(\Delta W,\Delta V)$ is transformed to be a nonlinear programming problem with linear inequalities constraints.

3.3 Bound of structural uncertainty

Section 3.2 discusses the bound of nonstructural noise uncertainty, which means only inequalities $|\Delta W|\le a$, $|\Delta V|\le b$ are necessary to stand instead of caring for the detailed form of $(\Delta W,\Delta V)$. Thus, the uncertainty $(\Delta W,\Delta V)$ may have infinite forms in structure. In this section, we will focus on giving the upper bound of structural noise uncertainty, and this work is also very useful for the actual producing process. Consider practical dynamic systems working in a complicated environment; this process must be disturbed by noises from different sources. Thus, for each noise source, there should be a limitation on noise uncertainty in order to meet the performance requirements. Suppose that all the noise sources are independent and noises are therefore uncorrelated; thus, the noise covariance matrix is as follows:

$$W^0=W+\Delta W=\begin{bmatrix}\sigma_1^2&0&\cdots&0\\0&\sigma_2^2&\cdots&0\\ \cdot&\cdot&\cdot&\cdot\\0&0&\cdots&\sigma_n^2\end{bmatrix}+\begin{bmatrix}\varsigma_1&0&\cdots&0\\0&\varsigma_2&\cdots&0\\ \cdot&\cdot&\cdot&\cdot\\0&0&\cdots&\varsigma_n\end{bmatrix}$$

$$V^0=V+\Delta V=\begin{bmatrix}\delta_1^2&0&\cdots&0\\0&\delta_2^2&\cdots&0\\ \cdot&\cdot&\cdot&\cdot\\0&0&\cdots&\delta_m^2\end{bmatrix}+\begin{bmatrix}e_1&0&\cdots&0\\0&e_2&\cdots&0\\ \cdot&\cdot&\cdot&\cdot\\0&0&\cdots&e_m\end{bmatrix}$$

Here $\sigma_s(s=1,2,\cdots,n)$ and $\delta_t(t=1,2,\cdots,m)$ are the entries of noise covariance matrix without uncertainty $(\Delta W, \Delta V)$. Non-negative parameters ς_s, e_t represent noise uncertainty.

Denote by $W_s(V_t)$ an $n\times n(m\times m)$ matrix whose $s-th(t-th)$ diagonal entry is 1 and other entries are zero:

$$W_s = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ \vdots & 0 & \cdots & 0 & \vdots \\ 0 & \cdots & 1_{ss} & \cdots & 0 \\ \vdots & 0 & \cdots & 0 & \vdots \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix}_{n\times n} \quad V_t = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ \vdots & 0 & \cdots & 0 & \vdots \\ 0 & \cdots & 1_{tt} & \cdots & 0 \\ \vdots & 0 & \cdots & 0 & \vdots \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix}_{m\times m} \quad (18)$$

According to Eq. (18), the noise uncertainty $(\Delta W, \Delta V)$ is

$$\Delta W = \sum_{s=1}^{n} \varsigma_s W_s \qquad \Delta V = \sum_{t=1}^{m} e_t V_t$$

By combining (13) and (15) together, there is $\forall i \in S$

$$\sum_{s=1}^{n} \varsigma_s tr(E_{si}) + \sum_{t=1}^{m} e_t tr(F_{ti}) \le r + \max_{j\in S} tr(Q_j) - tr(Q_i) \quad (19)$$

Where matrix $E_{si}, F_{ti} > 0, i\in S, s = 1,2,\cdots,n; t = 1,2,\cdots,m$ satisfies the following equations:

$$E_{si} = (\hat{A}_i - K_i\hat{C}_i)\sum_{j=1}^{N} p_{ij}E_{sj}(\hat{A}_i - K_i\hat{C}_i)^T + W_s \qquad s = 1,2,\cdots,n$$

$$F_{ti} = (\hat{A}_i - K_i\hat{C}_i)\sum_{j=1}^{N} p_{ij}F_{tj}(\hat{A}_i - K_i\hat{C}_i)^T + K_iV_tK_i^T \qquad t = 1,2,\cdots,m$$

In the following discussion, without the loss of generality, suppose that each column vector of filtering gain matrix K_i is non-zero. This fact can be easily shown by the following proof:

For any given $t_0 (1 \le t_0 \le m)$, if the $t_0 - th$ column vector of matrix K_i is zero, there is $K_i V_{t_0} K_i^T = 0$ which immediately results in $F_{t_0 i} = 0$; thus, there is no limitation or bound for parameter e_{t_0} because e_{t_0} has no contribution to the performance deviation ΔJ. In the subsequent part of this paper, only the bound of e_{t_0} with each column vector of K_i non-zero is considered.

Similarly, the search for an upper bound of structural noise uncertainty $(\Delta W, \Delta V)$ with inequality constraints (19) is equivalent to the following nonlinear programming problem:

$$\begin{aligned} & max \quad \varsigma_1 \cdot \varsigma_2 \ldots \varsigma_n \cdot e_1 \cdot e_2 \ldots e_m \\ & s.t. \sum_{s=1}^{n} \varsigma_s tr(E_{si}) + \sum_{t=1}^{m} e_t tr(F_{ti}) \le r + \max_{j\in S} tr(Q_j) - tr(Q_i) \\ & \varsigma_1, \varsigma_2, \cdots, \varsigma_n \ge 0 \qquad e_1, e_2, \cdots, e_m \ge 0 \qquad i \in S \end{aligned} \quad (20)$$

According to (17) and (20), the upper bound of both nonstructural and structural noise uncertainty could be deduced to solve a nonlinear programming problem with inequality constraints. In next section we will discuss the solution of this problem.

3.4 Analytical solution

$\Xi = \{(\Delta W, \Delta V)\}$ is a compact convex set while the inequalities in (17) and (20) compose a compact closed set on which function $a \cdot b$ and $\varsigma_1 \ldots \varsigma_n e_1 \ldots e_m$ are defined as continuous functions. Since continuous functions defined on a bounded closed set must have a maximum and minimum value, there exist the optimal solutions of $a, b, \varsigma_1 \ldots \varsigma_n, e_1 \ldots e_m$ to satisfy (17) and (20).

Decompose the original nonlinear programming problem (17) into N sub-problems:

$$\begin{aligned} & max \quad a_1 \cdot b_1 \\ & s.t. a_1 \cdot tr(D_1) + b_1 \cdot tr(G_1) \le r + \max_{j\in S} tr(Q_j) - tr(Q_1) \\ & max \quad a_2 \cdot b_2 \\ & s.t. a_2 \cdot tr(D_2) + b_2 \cdot tr(G_2) \le r + \max_{j\in S} tr(Q_j) - tr(Q_2) \\ & \qquad \vdots \\ & max \quad a_N \cdot b_N \\ & s.t. a_N \cdot tr(D_N) + b_N \cdot tr(G_N) \le r + \max_{j\in S} tr(Q_j) - tr(Q_N) \end{aligned}$$

By using the *Lagrange* method, the optimal analytical solution for each sub-problem is:

$$a_i^* = \frac{r + \max_{j\in S} tr(Q_j) - tr(Q_i)}{2tr(D_i)}$$
$$b_i^* = \frac{r + \max_{j\in S} tr(Q_j) - tr(Q_i)}{2tr(G_i)} \quad (21)$$

Thus, the analytical solution for the original nonlinear programming problem (17) is taken as

$$a^* = \min_{i\in S} a_i^* = \min_{i\in S}\{\frac{r + \max_{j\in S} tr(Q_j) - tr(Q_i)}{2tr(D_i)}\}$$
$$b^* = \min_{i\in S} b_i^* = \min_{i\in S}\{\frac{r + \max_{j\in S} tr(Q_j) - tr(Q_i)}{2tr(G_i)}\} \quad (22)$$

Similarly, for the analytical solution of problem (20), decompose it into N sub-problems:

$$\begin{aligned} & max \quad \varsigma_{11} \cdot \varsigma_{21} \cdots \varsigma_{n1} \cdot e_{11} \cdot e_{21} \cdots e_{m1} \\ & s.t. \quad \sum_{s=1}^{n} \varsigma_{s1} tr(E_{s1}) + \sum_{t=1}^{m} e_{t1} tr(F_{t1}) \le r + \max_{j\in S} tr(Q_j) - tr(Q_1) \end{aligned}$$

$$\begin{aligned}&max \quad \varsigma_{12}\cdot\varsigma_{22}\cdots\varsigma_{n2}\cdot e_{12}\cdot e_{22}\cdots e_{m2}\\ &s.t. \quad \sum_{s=1}^{n}\varsigma_{s2}tr(E_{s2})+\sum_{t=1}^{m}e_{t2}tr(F_{t2})\le r+\max_{j\in S}tr(Q_j)-tr(Q_2)\\ &\qquad\vdots\\ &max \quad \varsigma_{1N}\cdot\varsigma_{2N}\cdots\varsigma_{nN}\cdot e_{1N}\cdot e_{2N}\cdots e_{mN}\\ &s.t. \quad \sum_{s=1}^{n}\varsigma_{sN}tr(E_{sN})+\sum_{t=1}^{m}e_{tN}tr(F_{tN})\le r+\max_{j\in S}tr(Q_j)-tr(Q_N)\end{aligned}$$

By using the *Lagrange* method, the optimal analytical solution for each sub-problem is

$$\begin{aligned}&\varsigma_{si}^* = \frac{r+\max_{j\in S}tr(Q_j)-tr(Q_i)}{(n+m)tr(E_{si})} \quad s=1,2,\cdots,n,\\ &e_{ti}^* = \frac{r+\max_{j\in S}tr(Q_j)-tr(Q_i)}{(n+m)tr(F_{ti})} \quad t=1,2,\cdots,m,\ i=1,2,\cdots,N\end{aligned} \tag{23}$$

The analytical solution for the original nonlinear programming problem (20) is taken as

$$\begin{aligned}&\varsigma_s^* = \min_{i\in S}\varsigma_{si}^* = \min_{i\in S}\{\frac{r+\max_{j\in S}tr(Q_j)-tr(Q_i)}{(n+m)tr(E_{si})}\}\\ &e_t^* = \min_{i\in S}e_{ti}^* = \min_{i\in S}\{\frac{r+\max_{j\in S}tr(Q_j)-tr(Q_i)}{(n+m)tr(F_{ti})}\}\end{aligned} \tag{24}$$

Remark: The analytical solutions of the nonlinear programming problem (17), (20) are given by the above analysis; however, they are only optimal solutions for each sub-problem. These analytical solutions in Eq. (22) and (24) are local optimal but global sub-optimal. For a global optimal solution, only a numerical solution could be achieved by applying "fmincon" function in Matlab software. The optimal analytical solution of such nonlinear programming problem is still an open problem in mathematics for further exploration.

Theorem 2 Consider Markovian jump system (1) and (5). If the designer adopts state estimator (6) and Kalman filter gain (7), there exists a maximum admissible compact set Ξ . When the uncertainty of noise covariance matrix $(\Delta W,\Delta V)\in\Xi$, the deviation of system state estimation performance is ensured to be within the given precision r .

4. Mini-max robust filter

Let $K_1^*,K_2^*,\ldots,K_N^*$ denote the standard extended Kalman filtering gain corresponding to new noise covariance matrix $(W+\Delta W^*,V+\Delta V^*)$. According to the quality of Kalman filtering, there is

$$\Delta J(K_1^*,K_2^*,\cdots,K_N^*,\Delta W^*,\Delta V^*)\le\Delta J(K_1,K_2,\cdots,K_N,\Delta W^*,\Delta V^*)$$

On the other hand, for the same Kalman filtering gain $K_1^*,K_2^*,\cdots,K_N^*$, by applying Fact 1,

$$\begin{aligned}&\Delta J(K_1^*,K_2^*,\cdots,K_N^*,\Delta W,\Delta V)\le\\ &\Delta J(K_1^*,K_2^*,\cdots,K_N^*,\Delta W^*,\Delta V^*)\end{aligned}$$

Thus the saddle point inequality stands according to the above analysis:

$$\begin{aligned}&\Delta J(K_1^*,K_2^*,\cdots,K_N^*,\Delta W,\Delta V)\le\\ &\Delta J(K_1^*,K_2^*,\cdots,K_N^*,\Delta W^*,\Delta V^*)\le\\ &\Delta J(K_1,K_2,\cdots,K_N,\Delta W^*,\Delta V^*)\end{aligned} \tag{25}$$

By game theory, the optimal estimator under the worst situation is the mini-max estimator:

$$\begin{aligned}&\min_{K_i}\max_{(\Delta W,\Delta V)\in\Xi}\Delta J(K_1,K_2,\cdots,K_N,\Delta W,\Delta V)=\\ &\max_{(\Delta W,\Delta V)\in\Xi}\min_{K_i}\Delta J(K_1,K_2,\cdots,K_N,\Delta W,\Delta V)\end{aligned} \tag{26}$$

Remark: Traditional Kalman filtering design is performed on the basis that the noise covariance matrix is stationary and exactly known, and it will fail when the noise covariance matrix is unknown or has uncertainty. In our method, the filter design could be divided into two steps. First, design standard Kalman filter according to the stationary noise covariance matrix (W,V) , then via some technical methods such as noise control, the designer imposes the noise uncertainty to be within the given bound $(\Delta W^*,\Delta V^*)$, which could be presented in different forms (structural and nonstructural). In a practical dynamic process, the ideal deviation of performance under noise uncertainty is $\Delta J(K_1^*,K_2^*,\cdots,K_N^*,\Delta W,\Delta V)$. When the noise uncertainty reaches the maximum, the deviation of performance also reaches the maximum, which is $\Delta J(K_1^*,K_2^*,\cdots,K_N^*,\Delta W^*,\Delta V^*)$, and this deviation is less than the worst case $\Delta J(K_1,K_2,\cdots,K_N,\Delta W^*,\Delta V^*)\le r$. Saddle point inequality (25) means that as long as the noise uncertainty is within the admissible bound $(\Delta W^*,\Delta V^*)$, the deviation of performance for a practical process is less than given precision r . For this reason, the Kalman filter design has robustness over noise uncertainty and it is also a mini-max filter with Eq. (26) established.

5. Simulation

Consider the following two-mode discrete-time Markovian jump system:

Let the system mode $r_k=1$ be given by

$$\begin{aligned}&x_1(k+1)=0.5x_1(k)-0.2x_2(k)+0.02cos[x_1(k)+x_2(k)]+\omega_1^0\\ &x_2(k+1)=0.6x_2(k)+0.01sin[x_1(k)-x_2(k)]+\omega_2^0\end{aligned}$$

$$y(k) = x_1(k) + 0.5x_2(k) + \upsilon^0$$

Let the system mode $r_k = 2$ be given by

$$x_1(k+1) = -0.3x_1(k) + 0.1x_2(k) + 0.03 sin x_2(k) + \omega_1^0$$
$$x_2(k+1) = 0.1x_1(k) + 0.4x_2(k) + 0.01 cos x_1(k) + \omega_2^0$$
$$y(k) = x_1(k) + \upsilon^0$$

Where uncertain state and measurement noise is $\omega^0 = [\omega_1^0 \ \omega_2^0]^T$ and υ^0, and its stationary covariance matrix is known as $W = \begin{bmatrix} 0.4 & 0 \\ 0 & 0.5 \end{bmatrix}$, $V = 0.6$; system mode transition probability matrix is $P = \begin{bmatrix} 0.6 & 0.4 \\ 0.5 & 0.5 \end{bmatrix}$ the admissible bound of performance deviation is $r = 0.2$.

The detailed algorithm is as follows:

1) By applying Assumption 2, there is: $A_1 = \begin{bmatrix} 0.5 & -0.2 \\ 0.6 & 0 \end{bmatrix}$, $C_1 = [1 \ 0.5]$, $A_2 = \begin{bmatrix} -0.3 & 0.1 \\ 0.1 & 0.4 \end{bmatrix}$, $C_2 = [1 \quad 0]$, $H_{11} = [0.2 \quad 0.1]^T$, $E_1 = [0.1 \quad 0.1]$, $H_{12} = [0.15 \quad 0.1]^T$, $E_2 = [0.2 \quad 0.1]$, $H_{21} = H_{22} = 0$

2) Solve the equation Eq. (7), get Q_1, Q_2 and K_1, K_2 : $Q_1 = \begin{bmatrix} 0.9577 & 0.4102 \\ 0.4102 & 1.1021 \end{bmatrix}$, $Q_2 = \begin{bmatrix} 0.9856 & 0.5373 \\ 0.5373 & 1.2732 \end{bmatrix}$, $K_1 = \begin{bmatrix} 0.9484 \\ 1.1290 \end{bmatrix}$, $K_2 = \begin{bmatrix} 0.6956 \\ 0.7311 \end{bmatrix}$

3) Substitute the result to Eq. (17); by using the *Lagrange* method, the upper bound of nonstructural noise uncertainty is given as:

$$a^* = 0.1309, \qquad b^* = 0.1301$$

4) Let the new noise covariance matrix correspond to the maximum uncertainty:

$$W^* = W + \Delta W = W + a^* \cdot I_2,$$
$$V^* = V + \Delta V = V + b^* \cdot I_1$$

5) Repeat step 2), and the correspondent Q_1^*, Q_2^*, K_1^*, K_2^* for new noise covariance matrix (W^*, V^*) are:

$$Q_1^* = \begin{bmatrix} 1.0178 & 0.4417 \\ 0.4417 & 1.2142 \end{bmatrix}, \ Q_2^* = \begin{bmatrix} 1.1044 & 0.5845 \\ 0.5845 & 1.3097 \end{bmatrix},$$
$$K_1^* = \begin{bmatrix} 0.9877 \\ 1.2014 \end{bmatrix}, \ K_2^* = \begin{bmatrix} 0.7112 \\ 0.7633 \end{bmatrix}$$

6) With robust extended Kalman filtering applying, there is a saddle point inequality:

$$\Delta J(K_1^*, K_2^*, \Delta W, \Delta V) \le \Delta J(K_1^*, K_2^*, \Delta W^*, \Delta V^*)$$
$$= max\{tr(Q_1^*), tr(Q_2^*)\} - max\{tr(Q_1), tr(Q_2)\}$$
$$= 0.1553 < 0.2$$

7) Substitute the result of Q_1, Q_2, K_1, K_2 to Eq. (20); by using the *Lagrange* method, the upper bound of structural noise uncertainty is given as:

$$\varsigma_1^* = 0.2077, \qquad \varsigma_2^* = 0.1227, \qquad e_1^* = 0.0866$$

8) Let the new noise covariance matrix correspond to the maximum uncertainty:

$$W^* = W + \Delta W = W + diag\{\varsigma_1^*, \varsigma_2^*\},$$
$$V^* = V + \Delta V = V + e_1^*$$

9) Repeat step 2), and the correspondent Q_1^*, Q_2^*, K_1^*, K_2^* for new noise covariance matrix (W^*, V^*) are:

$$Q_1^* = \begin{bmatrix} 1.1098 & 0.4782 \\ 0.4782 & 1.2755 \end{bmatrix}, \ Q_2^* = \begin{bmatrix} 1.1401 & 0.6033 \\ 0.6033 & 1.2996 \end{bmatrix},$$
$$K_1^* = \begin{bmatrix} 1.0017 \\ 1.2234 \end{bmatrix}, \ K_2^* = \begin{bmatrix} 0.7208 \\ 0.7711 \end{bmatrix}$$

10) With robust extended Kalman filtering applying, there is a saddle point inequality:

$$\Delta J(K_1^*, K_2^*, \Delta W, \Delta V) \le \Delta J(K_1^*, K_2^*, \Delta W^*, \Delta V^*)$$
$$= max\{tr(Q_1^*), tr(Q_2^*)\} - max\{tr(Q_1), tr(Q_2)\}$$
$$= 0.1809 < 0.2$$

6. Conclusion

In this paper, a robust extended Kalman filter for a discrete-time Markovian jump nonlinear system under uncertain noise is considered. To maintain stability of a dynamic system when noise is “uncertain”, a new design method is given to obtain the maximum admissible bound of uncertainty to noise covariance matrix. The deviation of system estimation performance is thus guaranteed to be within a given precision. The noise uncertainty is in two different forms: nonstructural and structural. The analytical solution of the bound to noise uncertainty is also discussed in this paper, which is a global sub-optimal and conservative solution by using the *Lagrange* method. This work provides another way to achieve the optimal filter under the maximum noise uncertainty. The designer could first design the optimal filter by using stationary noise information and then impose the noise uncertainty to be within an admissible bound via noise control. It is proved that these two choices are equivalent by considering performance precision.

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